# VARIATIONAL SOLUTION OF A HEAT-CONDUCTION PROBLEM FOR A REGION WITH MOVING BOUNDARIES

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The Ainola variational principle is applied to a heat-conduction problem with moving boundaries.

Since the exact solution of a nonstationary heat-conduction problem for a region with moving boundaries is difficult, cumbersome, and inconvenient for practical calculations [1], we seek an approximate solution by using the variational formulation of L. Ya. Ainola [2] for the following boundary value problem:

$$\frac{1}{x^{m-1}}\frac{\partial}{\partial x}\left[x^{m-1}\lambda(x)\frac{\partial T(x,\tau)}{\partial x}\right]=c\rho(x)\frac{\partial T(x,\tau)}{\partial\tau}-q_v(x,\tau),$$
$$m=1,\ 2,\ 3,\quad s_1(\tau)<x<s_2(\tau),\quad \tau>0 \tag{1}$$

with the initial condition

$$T(x,0)=\varphi(x),\quad s_1(0)\leqslant x\leqslant s_2(0) \tag{2}$$

and the boundary conditions

$$T(x,\tau)|_{x=s_1(\tau)}=T(s_1(\tau),\tau)=T_{1c}(\tau),\quad \tau>0, \tag{3}$$
$$T(x,\tau)|_{x=s_2(\tau)}=T(s_2(\tau),\tau)=T_{2c}(\tau),\quad \tau>0, \tag{4}$$

where $\varphi(x)$, $T_{1c}(\tau)$, and $T_{2c}(\tau)$ are continuous functions satisfying the matching conditions

$$T_{1c}(0)=\varphi(s_1(0)),$$
$$T_{2c}(0)=\varphi(s_2(0)), \tag{5}$$

and $s_1(\tau)$ and $s_2(\tau)$ are continuous differentiable functions.

By introducing a new unknown function $u(x, \tau)$ such that

$$T(x,\tau)=u(x,\tau)+\varphi(x)+[T_{1c}(\tau)-\varphi(s_1(\tau))]\frac{s_2(\tau)-x}{s_2(\tau)-s_1(\tau)}+[T_{2c}(\tau)-\varphi(s_2(\tau))]\frac{x-s_1(\tau)}{s_2(\tau)-s_1(\tau)},$$

we reduce problem (1)-(4) to a problem with zero initial and boundary conditions:

$$\frac{1}{x^{m-1}}\frac{\partial}{\partial x}\left[x^{m-1}\lambda(x)\frac{\partial u(x,\tau)}{\partial x}\right]-c\rho(x)\frac{\partial u(x,\tau)}{\partial\tau}-f(x,\tau)=0, \tag{6}$$

$$m=1,\ 2,\ 3,\quad s_1(\tau)<x<s_2(\tau), \tag{1'}$$
$$u(x,0)=0,\quad s_1(0)\leqslant x\leqslant s_2(0), \tag{2'}$$
$$u(x,\tau)=u(s_1(\tau),\tau)=0,\quad \tau>0, \tag{3'}$$
$$u(x,\tau)=u(s_2(\tau),\tau)=0,\quad \tau>0, \tag{4'}$$

where

$$f(x,\tau)=\frac{c\rho(x)}{s_2(\tau)-s_1(\tau)}\Big\{[T'_{1c}(\tau)-\varphi'(s_1(\tau))s'_1(\tau)][s_2(\tau)-x]$$
$$+[T'_{2c}(\tau)-\varphi'(s_2(\tau))s'_2(\tau)][x-s_1(\tau)]+[T_{1c}(\tau)-T_{2c}(\tau)$$

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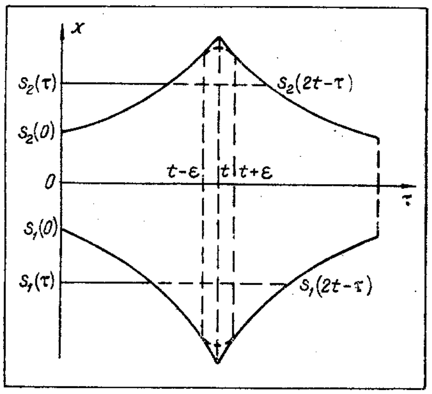


Fig. 1. Domain with moving boundaries in the x-$\tau$ phase plane.

$$+\varphi(s_2(\tau))-\varphi(s_1(\tau))]\,\frac{[x-s_1(\tau)]\,s_2'(\tau)+[s_2(\tau)-x]\,s_1'(\tau)}{s_2(\tau)-s_1(\tau)}\Bigg\}$$

$$-\frac{1}{x^{m-1}}\left\{[\lambda(x)\,x^{m-1}\varphi'(x)]'+\frac{T_{2c}(\tau)-T_{1c}(\tau)+\varphi(s_1(\tau))-\varphi(s_2(\tau))}{s_2(\tau)-s_1(\tau)}\times[\lambda(x)\,x^{m-1}]'\right\}-q_v(x,\ \tau).$$

We extend the domain D of the phase plane bounded by the characteristics $\tau = 0$ and $\tau = t$ and the curves $x = s_1(\tau)$ and $x = s_2(\tau)$ to domain $\bar{D}$ bounded by the characteristics $\tau = 0$ and $\tau = 2t$ and the curves $x = \bar{s}_1(\tau)$ and $x = \bar{s}_2(\tau)$ (Fig. 1) such that

$$\bar{s}_1(\tau)=\begin{cases}s_1(\tau), & 0\leqslant\tau\leqslant t,\\ s_1(2t-\tau), & t<\tau\leqslant 2t,\end{cases}\tag{7}$$

$$\bar{s}_2(\tau)=\begin{cases}s_2(\tau), & 0\leqslant\tau\leqslant t,\\ s_2(2t-\tau), & t<\tau\leqslant 2t.\end{cases}\tag{8}$$

Then the following statement is true in domain $\bar{D}$: if $u(x,\ \tau)$ is a solution of problem (1')-(4') for $\bar{s}_1(\tau) < x < \bar{s}_2(\tau)$, $0 < \tau < 2t$, the functional

$$J(u)=\iint\limits_{\bar{D}}\left\{\frac{1}{x^{m-1}}\frac{\partial}{\partial x}\left[x^{m-1}\lambda(x)\frac{\partial u(x,\ \tau)}{\partial x}\right]-c\rho(x)\frac{\partial u(x,\ \tau)}{\partial\tau}-2f(x,\ \tau)\right\}x^{m-1}u(x,\ 2t-\tau)\,dx\,d\tau\tag{9}$$

has a stationary value; i.e., $\delta J(u) = 0$.

This is easy to prove by starting with the equality

$$\iint\limits_{\bar{D}}F(x,\ \tau)\,G(x,\ 2t-\tau)\,dx\,d\tau=\iint\limits_{\bar{D}}F(x,\ 2t-\tau)\,G(x,\ \tau)\,dx\,d\tau,$$

which is valid for any functions F and G which are continuous in $\bar{D}$, for any condition (2'), and conditions (3') and (4') for $x = \bar{s}_i(\tau)$, $i = 1,\ 2$. We note that $\bar{s}_i(\tau)$ may not be differentiable at $\tau = t$. Then in the interval $t-\varepsilon < \tau < t + \varepsilon$, where $\varepsilon > 0$ is a sufficiently small number, $\bar{s}_i(\tau)$ can be smoothed so that it will be differentiable for all $0 < \tau < 2t$, as shown by the dashed line in Fig. 1. Moreover, we are interested in values of $u(x,\ \tau)$ only in domain D, i.e., for $0 < \tau < t$, and values of $t > 0$ have no practical significance.

As an example we consider the use of the functional (9) to obtain an approximate analytical solution of the symmetrical problem of the temperature distribution in homogeneous one-dimensional bodies (plate, cylinder, sphere) without heat sources for a constant initial temperature and for a constant temperature on a moving boundary. In this case the boundary value problem (1)-(4) has the form

$$\frac{1}{x^{m-1}}\frac{\partial}{\partial x}\left[x^{m-1}\frac{\partial T(x,\ \tau)}{\partial x}\right]=\frac{1}{a}\frac{\partial T(x,\ \tau)}{\partial\tau},\tag{1''}$$

$$m=1,\ 2,\ 3,\quad 0<x<s(\tau),\quad \tau>0,\tag{2''}$$

$$T(0,\ x)=T_0,\quad 0\leqslant x\leqslant s(0),\qquad T(s(\tau),\ \tau)=T_c,\quad \tau>0,\tag{3''}$$

$$\frac{\partial T(0,\ \tau)}{\partial x}=0,\quad \tau>0\tag{4''}$$

To satisfy the matching condition (5) we set

$$T(s(\tau),\tau)=T_c(\tau)=\begin{cases}\dfrac{T_0-T_c}{\tau_0^2}(\tau-\tau_0)^2+T_c, & 0\leqslant\tau\leqslant\tau_0,\\ T_c, & \tau_0\leqslant\tau.\end{cases}$$

Further, in accord with (6), we introduce a function $u(x,\ \tau)$ such that

$$T(x,\ \tau)=u(x,\ \tau)+T_c(\tau).$$

As a result the boundary value problem (1")-(4") becomes:

$$\frac{1}{x^{m-1}}\frac{\partial}{\partial x}\left[x^{m-1}\frac{\partial u}{\partial x}\right]=\frac{1}{a}\frac{\partial u}{\partial\tau}+\frac{T_c'(\tau)}{a},\tag{1'''}$$

$$m=1,\ 2,\ 3,\quad 0<x<s(\tau),\quad \tau>0,$$

$$u(x,\ 0)=0, \quad 0 \leqslant x \leqslant s(0), \tag{2'''}$$

$$u(s(\tau),\ \tau)=0, \quad \tau>0, \tag{3'''}$$

$$\frac{\partial u(0,\ \tau)}{\partial x}=0, \ \tau>0, \tag{4'''}$$

and the functional (9) can be written as

$$J(u)=\int_0^{2t}\int_0^{\bar{s}(\tau)}\left\{\frac{1}{x^{m-1}}\frac{\partial}{\partial x}\left[x^{m-1}\frac{\partial u(x,\ \tau)}{\partial x}\right]-\frac{1}{a}\frac{\partial u(x,\ \tau)}{\partial \tau}-\frac{T_c'(\tau)}{a}\right\}x^{m-1}u(x,\ 2t-\tau)\,dx\,d\tau, \tag{9'}$$

where, as indicated above

$$\bar{s}(\tau)=\begin{cases}s(\tau), & 0\leqslant\tau\leqslant t,\\ s(2t-\tau), & t\leqslant\tau\leqslant 2t.\end{cases} \tag{7'}$$

Following Kantorovich [3] we set

$$u(x,\ \tau)=\left[1-\frac{x^2}{\bar{s}^2(\tau)}\right]\psi(\tau),$$

where $\psi(\tau)$ is an unknown function satisfying the condition $\psi(0) = 0$. Integrating (9') with respect to x gives

$$J=-\frac{8}{a(m+2)(m+4)m}\int_0^{2t}\left\{\frac{m(m+4)}{2}\left[\frac{a}{\bar{s}^2(\tau)}-\frac{\bar{s}'(\tau)}{(m+4)\bar{s}(\tau)}\right]\psi(\tau)\right.$$

$$\left.+\psi'(\tau)+\frac{m+4}{2}T_c'(\tau)\right\}\bar{s}^m(\tau)\psi(2t-\tau)\,d\tau.$$

Taking account of the fact that $\bar{s}(\tau) = \bar{s}(2t-\tau)$ Euler's equation for this functional will be

$$\psi'(\tau)+\frac{m(m+4)a}{2\bar{s}^2(\tau)}\psi(\tau)=-\frac{m+4}{4}T_c'(\tau).$$

For the $\psi(\tau)$ of interest to us, when $0 < \tau < t$, we find by using (7') that the condition for J to be stationary is given by the equation

$$\psi'(\tau)+\frac{m(m+4)a}{2s^2(\tau)}\psi(\tau)=-\frac{m+4}{4}T_c'(\tau), \tag{10}$$

whose solution is

$$\psi(\tau)=-\frac{(m+4)(T_0-T_c)}{2\tau_0^2}\exp\left[-\frac{m(m+4)a}{2}\int_0^{\tau}\frac{dz}{s^2(z)}\right]$$

$$\times\begin{cases}\displaystyle\int_0^{\tau}(z-\tau_0)\exp\left[\frac{m(m+4)a}{2}\int_0^{z}\frac{dy}{s^2(y)}\right]dz, & 0\leqslant\tau\leqslant\tau_0,\\ \displaystyle\int_0^{\tau}(z-\tau_0)\exp\left[\frac{m(m+4)a}{2}\int_0^{z}\frac{dy}{s^2(y)}\right]dz, & \tau_0\leqslant\tau.\end{cases} \tag{11}$$

By going to the limit $\tau_0 \to 0$ in (11) we obtain the first approximation to the solution of problem (1''')-(4''') in the form

$$\theta=\frac{T(x,\ \tau)-T_c}{T_0-T_c}=\frac{m+4}{4}\left[1-\frac{x^2}{s^2(\tau)}\right]\exp\left\{-\frac{m(m+4)a}{2}\int_0^{\tau}\frac{dz}{s^2(z)}\right\}. \tag{12}$$

In the special case of a boundary moving uniformly from $s(0) = l_0$, i.e., for $s(\tau) = v\tau + l_0$, we have

$$\theta=\frac{m+4}{4}\left[1-\frac{x^2}{(v\tau+l_0)^2}\right]\exp\left[-\frac{m(m+4)a\tau}{2l_0(v\tau+l_0)}\right]. \tag{12'}$$

To obtain the second approximation we set

$$u(x,\ \tau)=\left[1-\frac{x^2}{\bar{s}^2(\tau)}\right]\psi_1(\tau)+\left[1-\frac{x^2}{\bar{s}^2(\tau)}\right]\frac{x^2}{\bar{s}^2(\tau)}\psi_2(\tau),$$

where $\psi_1(\tau)$ and $\psi_2(\tau)$ are unknown functions satisfying the condition $\psi_1(0) = \psi_2(0) = 0$.

The condition for the functional J to be stationary is given by the following system of Euler equations:

$$\psi_1'(\tau) + \frac{ma}{12s^2(\tau)}[(4-m)(m+6)\psi_1(\tau) - (m^2+6m+32)\psi_2(\tau)] = \frac{(m-4)(m+6)}{24} T_c'(\tau),$$

$$\psi_2'(\tau) + \frac{(m+6)(m+8)a}{12s^2(\tau)}[m\psi_1(\tau) + (m+4)\psi_2(\tau)] = -\frac{(m+6)(m+8)}{24} T_c'(\tau).$$

The solution of this system of first order equations with variable coefficients is [4]

$$\psi_1(\tau) = \frac{\beta_2 z_1(\tau) - \beta_1 z_2(\tau) + m(4-m)(m+6)[z_2(\tau) - z_1(\tau)]}{m(m+6)(m+8)(\beta_2-\beta_1)},$$

$$\psi_2(\tau) = \frac{z_2(\tau) - z_1(\tau)}{\beta_2 - \beta_1},$$

where

$$z_i(\tau) = -\frac{(m+6)(m+8)}{24}\beta_i \exp\left\{-\frac{a\beta_i}{12}\int_0^\tau \frac{dy}{s^2(y)}\right\}\int_0^\tau T_c'(y)\exp\left\{\frac{a\beta_i}{12}\int_0^y \frac{d\xi}{s^2(\xi)}\right\}dy,$$

$$\beta_i = 4[2(m+2)(m+6) + (-1)^i\sqrt{(m+2)(m+6)(m^2+8m+48)}],$$
$$i = 1,\ 2.$$

In the limit $\tau_0 \to 0$

$$z_i(\tau) = \frac{(m+6)(m+8)}{24}\beta_i \exp\left\{-\frac{a\beta_i}{12}\int_0^\tau \frac{dy}{s^2(y)}\right\}.$$

Thus the second approximation to the solution of problem (1‴)-(4‴) has the form

$$\theta = \frac{T(x,\tau) - T_c}{T_0 - T_c} = \frac{1 - \dfrac{x^2}{s^2(\tau)}}{192m\sqrt{(m+2)(m+6)(m^2+8m+48)}}$$
$$\times\left\{[\beta_2 - m(4-m) - m(m+6)(m+8)x^2]\beta_1\exp\left(-\frac{a\beta_1}{12}\int_0^\tau\frac{dy}{s^2(y)}\right)\right. \qquad (13)$$
$$\left.-[\beta_1 - m(4-m)(m+6) - m(m+6)(m+8)x^2]\beta_2\exp\left(-\frac{a\beta_2}{12}\int_0^\tau\frac{dy}{s^2(y)}\right)\right\}.$$

In conclusion we note that the method described here for solving a heat-conduction problem for a region with moving boundaries can be used for other boundary conditions also.

## NOTATION

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|---|---|
| $T(x, \tau)$ | is the running temperature; |
| $\theta = (T(x, \tau)-T_c)/(T_0-T_c)$ | is the dimensionless relative temperature; |
| $T_0$ | is the initial temperature; |
| $T_c$ | is the temperature of the medium; |
| $l_0 = s(0)$ | is the initial position of the moving boundary; |
| $x = s(\tau)$ | is the running position of the moving boundary; |
| $x$ | is the coordinate of a point of the body; |
| $\tau$ and $t$ | are values of the time; |
| $a = \lambda/c\rho$ | is the thermal diffusivity; |
| $\lambda$ | is the thermal conductivity; |
| $c\rho$ | is the volumetric heat capacity of the body; |
| $q_v(x, \tau)$ | is the volumetric heat-release rate. |

## LITERATURE CITED


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